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LETTER TO THE EDITOR

# Degenerations of Sklyanin algebra and Askey–Wilson polynomials

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**Abstract.** A new trigonometric degeneration of the Sklyanin algebra is found and the functional realization of its representations in space of polynomials in one variable is studied. A further contraction gives the standard quantum algebra $U_q(sl(2))$. It is shown that the degenerate Sklyanin algebra contains a subalgebra isomorphic to algebra of functions on the quantum sphere $(SU(2)/SO(2))_{q^{1/2}}$. The diagonalization of general quadratic form in its generators leads in the functional realization to the difference equation for Askey–Wilson polynomials.

Sklyanin algebra (SA) [1] plays a very important role in the theory of quantum integrable systems being the algebra of observables in the models with elliptic $R$-matrix. It is the algebra with quadratic relations that follow from the Yang–Baxter equation for quantum monodromy matrices. The trigonometric degeneration of the $R$-matrix gives rise to a number of the well known exactly solvable models like $XXZ$ spin chain, sine-Gordon, etc. In this case the SA degenerates [2] to the quantum group $U_q(sl(2))$—deformation of the universal enveloping algebra of $sl(2)$ introduced by Kulish and Reshetikhin [3] and then extensively studied in more general mathematical context [4].

In this letter, we show that $U_q(sl(2))$ is a very particular case of a more general family of trigonometric degenerations of the SA. More precisely, the standard $U_q(sl(2))$ can be obtained as further contraction of the degenerate SA. On the other hand, we show that the trigonometric limit of SA contains a subalgebra isomorphic to $Fun_{q^{1/2}}(SU(2)/SO(2))$—the algebra of functions on the quantum sphere $(SU(2)/SO(2))_{q^{1/2}}$. The generators of this algebra can be realized as difference operators acting on polynomials in one variable. This realization follows directly from the representation theory of SA [1]. We argue that the diagonalization problem for general quadratic form in the generators (which is commonly interpreted as the Hamiltonian of a proper physical system) is equivalent to the difference Hamiltonian of a proper physical system) is equivalent to the difference equation for the Askey–Wilson polynomials [5]. In particular, the diagonalization of the '$SO(2)$-invariant' element of $Fun_{q^{1/2}}(SU(2)/SO(2))$ (i.e. the generator of the commutative algebra $Fun_{q^{1/2}}(SO(2)\backslash SU(2)\backslash SO(2))$ leads to the Rogers–Askey–Ismail polynomials [6]. At the same time these polynomials are known to give zonal spherical functions on the quantum spheres [7, 8].

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The SA is the two-parametric deformation of $U_q(gl(2))$. It is the algebra with four generators $S_0$, $S_1$, $S_2$, $S_3$ and the following quadratic relations [1]:

$$\begin{aligned} &[S_0, S_\lambda] = \mathrm{i}J_{\mu\nu}(S_\mu, S_\nu) \\ &[S_\lambda, S_\mu] = \mathrm{i}(S_0, S_\nu). \end{aligned} \tag{1}$$

Here [,] and (,) are commutator and anticommutator respectively and $(\lambda, \mu, \nu)$ stands for any cyclic permutation of (1, 2, 3). The structure constants $J_{\mu\nu}$ are parametrized as follows: $J_{\mu\nu} = (J_\mu - J_\lambda)/J_\nu$ where

$$\begin{aligned} &J_1 = \theta_4(2\gamma)\theta_4(0)/(\theta_4(\gamma))^2 \\ &J_2 = \theta_3(2\gamma)\theta_3(0)/(\theta_3(\gamma))^2 \\ &J_3 = \theta_2(2\gamma)\theta_2(0)/(\theta_2(\gamma))^2. \end{aligned} \tag{2}$$

We use the standard notation [9] for the theta functions with elliptic nome $h = \exp(\mathrm{i}\pi\tau)$. The two deformation parameters are just $h$ and $\gamma$; they are supposed to be real. The two central (Casimir) elements are given by

$$\begin{aligned} &K_0 = (S_0)^2 + (S_1)^2 + (S_2)^2 + (S_3)^2 \\ &K_1 = J_1(S_1)^2 + J_2(S_2)^2 + J_3(S_3)^2. \end{aligned} \tag{3}$$

Trigonometric degeneration of the SA can be obtained by tending $h$ to 0. To do this let us redefine the generators as follows:

$$\begin{aligned} &A = \mathrm{i}h^{-1/4}(S_0 - \tanh(\pi\gamma)S_3)/(4\sinh(\pi\gamma)) \\ &D = \mathrm{i}h^{-1/4}(S_0 + \tanh(\pi\gamma)S_3)/(4\sinh(\pi\gamma)) \\ &C = \mathrm{i}h^{\pm 1/4}(S_1 - \mathrm{i}S_2)/(2\sinh(2\pi\gamma)) \\ &B = \mathrm{i}h^{-3/4}(S_1 + \mathrm{i}S_2)/(8\sinh(2\pi\gamma)). \end{aligned} \tag{4}$$

We consider only the representations of series (a) in terminology [1]. It follows from the realization of the generators of SA by difference operators [1] that these combinations have a finite limit when $h$ goes to 0. Substituting (4) into (1) and denoting $q = \exp(-2\pi\gamma)$ one obtains the relations in the degenerate SA:

$$DC = qCD \qquad CA = qAC \tag{5}$$

$$[A, D] = (1/4)(q - q^{-1})^3 C^2 \tag{6}$$

$$[B, C] = (A^2 - D^2)/(q - q^{-1}) \tag{7}$$

$$AB - qBA = qDB - BD = -(1/4)(q^2 - q^{-2})(DC - CA). \tag{8}$$

We would like to stress that though $J_{12} = 0$ when $h = 0$ and naively $A$ and $D$ would commute the careful limiting procedure taking into account the order in $h$ of each term in (1) gives just the non-vanishing commutator (6).

According to the general representation theory of the SA developed in [1] the finite dimensional irreducible representations of the limiting algebra (5)–(8) are parametrized by a non-negative integer or half-integer number $j$ (spin of the representation). The spin-$j$ representation). The spin-$j$ representation can be realized in $(2j+1)$-dimensional space of polynomials of degree $2j$ by the following difference operators:

$$C = \left(\frac{2}{(q - q^{-1})}\right)\left(\frac{1}{(z - z^{-1})}\right)(T_+ - T_-) \tag{9}$$

$$A = q^{-j}\left(\frac{1}{(z-z^{-1})}\right)(zT_+ - z^{-1}T_-) \tag{10}$$

$$D = q^{j}\left(\frac{1}{(z-z^{-1})}\right)(zT_- - z^{-1}T_+) \tag{11}$$

$$B = \left(\frac{1}{(2(q-q^{-1})}\right)\left(\frac{1}{(z-z^{-1})}\right)(q^{2j}(z^2T_- - z^{-2}T_+) - q^{-2j}(z^2T_+ - z^{-2}T_-) - (q+q^{-1})(T_+ - T_-)) \tag{12}$$

where $T_+f(z) = f(qz)$, $T_-f(z) = f(q^{-1}z)$. In the classical limit $q=1$ (9)–(12) give the standard functional realization of the $sl(2)$-generators $C$, $B$ and $H=(D-A)/(4\pi\gamma)$ by differential operators.

It is remarkable that the standard quantum algebra $U_q(sl(2))$ can be obtained from (5)–(8) by the contraction procedure: $B$ to $B$, $A$ to $\varepsilon A$, $D$ to $\varepsilon D$, $C$ to $\varepsilon^2 C$; $\varepsilon$ tends to zero. For the representation (9)–(12) the contraction effectively means that $z$ goes to infinity and we can formally put $z^{-1}$ equal to 0 thus obtaining the well known realization of $U_q(sl(2))$ in space of holomorphic functions.

One can see from (3) that the trigonometric limit of the central element $K_0 - K_1$ is proportional to $AD + (1/4q)(q-q^{-1})^2C^2$. It follows from (9)–(12) that the value of this Casimir element does not depend on $j$ and is equal to 1. Therefore, this operator can be put equal to 1:

$$AD + \frac{1}{(4q)}(q-q^{-1})^2C^2 = 1. \tag{13}$$

This relation is to be added to (5)–(8).

Note that $A$, $D$, $C$ generate the subalgebra with the relations (5), (6), (13). Remarkably enough, it is isomorphic to the algebra of functions on the quantum sphere $(SU(2)/SO(2))_{q^{1/2}}$ [8] (note this change of $q$ to $q^{1/2}$!). Let us point out that the generator $A+D$ (*the former* $S_0$) corresponds under this isomorphism to the simplest non-trivial element of $Fun_{q^{1/2}}(Su(2)/SO(2)) = F_{q^{1/2}}$ invariant with respect to the quasiregular action of the proper 'twisted primitive elements' [7] of $U_{q^{1/2}}(su(2))$ representing the 'infinitesimal $SO(2)$-rotation'. In other words, the (commutative) algebra of functions on the double quantum coset space $Fun_{q^{1/2}}(SO(2)\backslash SU(2)SO(2))$ is generated by 1 and $A+D$. The operators (9)–(11) form a (reducible) representation of $F_{q^{1/2}}$.

Now let us turn to the problem of diagonalization of a quadratic form in generators of $F_{q^{1/2}}$ in the space of spin-$j$ representation of the whole algebra (5)–(8). Such quadratic operator could be interpreted as the Hamiltonian of a proper physical system. It is known [10, 11] that in the classical $sl(2)$ case all exactly solvable problems of quantum mechanics on the line can be obtained in this way.

Consider the general quadratic form in $A$, $D$, $C$:

$$Q(r,k;\alpha,\beta) = q^rA^2 + q^{-r}D^2 - \tfrac{1}{4}(q^{2k+1} + q^{-2k-1})(q-q^{-1})^2C^2 + (q-q^{-1})(\alpha AC + \beta DC) \tag{14}$$

where $r$, $k$, $\alpha$, $\beta$ are arbitrary parameters. Substituting (9)–(11) into (14) we obtain the following eigenvalue equation for Laurent polynomials in $z$ $P_m(z)$:

$$Q(r,k;\alpha,\beta)P_m(z) = E_mP_m(z). \tag{15}$$

One can see that when $E_m = q^{2(j-m)-r} + q^{-2(j-m)+r}$ (15) coincides with the difference equation for Askey–Wilson polynomials $P_m((z+z^{-1})/2; a, b, c, d|q^2)$ $(=P_m(z)$ for brevity)

$$A(z)(P_m(q^2z) - P_m(z)) + A(z^{-1})) + A(z^{-1})(P_m(q^{-2}z) - P_m(z)) = (q^{-2m} - 1)(1 - abcdq^{2m-1})P_m(z) \tag{16}$$

where $A(z) = (1-az)(1-bz)(1-cz)(1-dz)/((1-z^2)(1-q^2z^2))$ and the parameters $a$, $b$, $c$, $d$ are expressed through $r$, $k$, $\alpha$, $\beta$ as follows:

$$(1-az)(1-bz)(1-cz)(1-dz) = 1 + 2\beta q^{r+1-j} - (q^{2k+1} + q^{-2k-1})q^{r+1-2j}z^2 + 2\alpha q^{r+1-3j}z^3 + q^{2r+2-4j}z^4. \tag{17}$$

In particular, for $r = k = \alpha = \beta = 0$ we have $Q(0, 0, 0, 0) = (A+D)^2 - 2$ and (16) reduces to the equation for the Rogers–Askey–Ismail polynomials $C_m((z+z^{-1})/2; q^{-2j}|q^2)$ [6] which are also known as Macdonald's polynomials for the root system $A_1$ [12]. So the eigenfunctions of the $SO(2)$-invariant element of $F_{q^{1/2}}$ discussed above are expressed in terms of these polynomaials. We note also that at the same time the Askey–Wilson polynomials (with particular values of the parameters) provide the most general family of zonal spherical functions on the quantum group $U_q(su(2))$ [7].

In conclusion let us make a few remarks. The diagonalization of a more general quadratic form in all the generators including $B$ leads to a more complicated eigenvalue difference equation corresponding to the so-called quasi-exactly solvable problems [11] in the classical limit. In case of the contraction to $U_q(sl(2))$ the little Jacobi polynomials appear.

The role of the quadratic algebra (5)–(8) (obtained as a result of the limiting procedure from the SA) in integrable models with trigonometric $R$-matrix is not clear enough at the moment. However, as it was shown in [13] the scattering of dressed excitations in the $XXZ$ antiferromagnetic spin chain can be described in terms of zonal spherical elements of the algebra of functions on the quantum sphere and/or hyperboloid.

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